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A boundary-field integral equation for analysis of cavity acoustic spectrum

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Abstract

The acoustic spectrum of cavities can be identified using integral equation formulations. Because of the transcendental dependence on frequency of the Green function, difficulties arise in calculating acoustic frequencies and modes of vibration when the Kirchhoff–Helmholtz boundary-integral operator is applied. This trouble is circumvented by the present, nonstandard, integral formulation that, by using the fundamental solution of the Laplace operator, allows the identification of acoustic spectra of cavities through solution of a standard eigenvalue problem. This formulation is compared both with that based on the Kirchhoff–Helmholtz operator and with an alternative integral approach introduced in the past that, akin to the one used here, analyzes cavity acoustics in terms of an eigenvalue problem. The numerical investigation deals both with a simple box-shaped cavity and with cavities related to applications of aeronautical interest.

Keywords: Cavity acoustics; Acoustic spectrum; Integral formulations

1. Introduction

The acoustic characterization of arbitrarily shaped cavities represents a crucial aspect in several design applications. For transportation vehicles (like, e.g., aircraft, cars, trains) its importance is not only related to passenger comfort in cabin, but also to the dynamic response of elastic structures interacting with the pressure field arising around them. For instance, this problem is of great importance for payload inside launcher fairings. Indeed, during launcher take-off and atmospheric climb, a payload could be damaged by the acoustic field generated inside the fairing by wall structure vibrations; such structure vibrations may be induced, for example, by the exterior pressure field.

In the past, several methods have been proposed to address the problem. Each of them presents advantages and drawbacks from the numerical point of view, that depend upon the shape of the acoustic cavity and/or upon the frequency range of interest [extensive reviews of this subject have been presented by Dowell (1980) and Crighton et al. (1992)]. Among these, one of the most widely used, especially when dealing with structural–acoustic coupling, is the modal approach [see, for instance, Dowell et al. (1977) and Gennaretti and Iemma (2003)] in which the acoustic field is expressed by a linear combination of the natural modes of vibration of the enclosure (i.e., the eigenfunctions of the Laplace operator). Its effectiveness and applicability depend upon the knowledge of the acoustic free modes of

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vibration that have a significant influence in the frequency range of interest. For cavities of arbitrary shape, the acoustic natural modes of vibration are not known analytically, and therefore must be determined through the application of suitable numerical and/or experimental procedures. However, the accuracy of these approximated modes rapidly decreases as the associated eigenfrequency increases. This fact makes the modal approach not particularly attractive for applications concerning the higher part of the spectrum. In addition, even when the analysis is limited to the low/medium frequency range, the evaluation of mode shapes and eigenfrequencies is often an expensive and time-consuming task.

The present work is mostly of theoretical nature and its aim is to present a novel numerical method for acoustic modal identification that is based on a nonstandard integral equation formulation. Before entering into details, it is worth recalling the difficulties and uncertainties that arise when using the classical Kirchhoff–Helmholtz boundary integral operator in modal acoustic analysis. For an acoustic enclosure with homogeneous Neumann-type boundary conditions, its numerical solution yields a homogeneous set of frequency-dependent equations. The roots of the corresponding characteristic equation are the approximated eigenvalues of the original spatial differential operator (i.e., the acoustic natural frequencies of vibration), whereas the associated eigensolutions represent an approximation of its eigenfunctions (i.e., the acoustic natural modes of vibration). Unfortunately, because of the presence of the Green function in the Kirchhoff–Helmholtz operator (determined as the fundamental solution of the wave equation in an unbounded domain), this characteristic equation is a transcendental function of frequency, and finding its roots is a challenging task. This problem is solved by using suitable numerical methodologies that, typically, either employ iterative techniques [among them, the secant method (Tai and Shaw, 1974) and the null eigenvalue seeking (Lu and Yau, 1991)], or are based on local approximations of the coefficients of the discretized Kirchhoff–Helmholtz integral operator (Kirkup and Amini, 1993). However, none of these methodologies yield reliable results in those frequency ranges where a high modal density is present. In addition, they require a significant computational effort for problems involving a large number of unknowns.

An alternative approach for the identification of the acoustic spectrum of cavities has been introduced by Succi (1987). This approach is based on knowledge of the eigenfunctions of the Laplace operator, with homogeneous Neumann-type boundary conditions, in a (simple-shaped) domain enclosing the cavity under examination. These eigenfunctions are used in finite expansions for expressing both the cavity acoustic natural modes of vibration and the Green function associated to the wave equation with homogeneous Neumann-type boundary conditions in the outer domain. Then, application of the boundary integral equation approach, together with linear independence of the Laplace operator eigenfunctions, yield a standard eigenproblem. The eigenvalues represent the approximated acoustic natural frequencies of vibration of the cavity and the eigenvectors represent the coefficients of the eigenfunction expansions that express the acoustic natural modes of vibration. Although giving a simplified approach for the numerical evaluation of cavity acoustics, this method introduces some inaccuracy associated to the fact that the cavity acoustic modes are defined through functions that satisfy the homogeneous boundary conditions in the outer domain.

Akin to Succi's method, the aim of the approach presented here is to identify the acoustic spectrum of cavities by solving a standard eigenproblem. The transcendental dependence on frequency of the integral representation of the acoustic field is avoided by assuming the Green function to be the fundamental solution of the Laplace operator in an arbitrary domain enclosing the cavity. Then, the acoustic natural modes of vibration are approximated through finite expansions of arbitrary linearly independent functions that satisfy the homogeneous Neumann boundary conditions in the cavity domain, and the final eigenproblem for the cavity acoustic analysis is obtained by applying the Galërkin approach. This procedure eliminates the inaccuracies arising in the Succi method and assures the convergence of the solution to the free vibration modes of the cavity. The drawback is that the integral representation is not limited to the surface of the cavity boundary, in that it includes integrals evaluated in the volume enclosed by the cavity under examination.

The following sections discuss and compare the above three approaches for the analysis of cavity acoustics spectra. Specifically, Section 2 introduces the cavity acoustics problem and outlines the solution provided by the approach based on the classical Kirchhoff–Helmholtz integral formulation. Section 3 describes the interior acoustics solution suggested by Succi (1987), whereas the formulation proposed here is presented in Section 4. Section 5 starts presenting the results of a numerical investigation that concerns a simple box-shaped cavity. It analyzes the advantages and disadvantages of the three approaches discussed above comparing the convergence behavior of their solutions, and hence providing an estimate of the capabilities of the formulation introduced here. Then, the methodology introduced in this paper is applied to the analysis of cavities related to problems of aeronautical interest and the results obtained are compared to predictions given by a finite-element solution.

2. The Kirchhoff–Helmholtz equation approach

This section outlines an approach for the identification of cavity acoustic spectra based on the Kirchhoff–Helmholtz boundary integral equation. This integral formulation is discussed in order to illustrate the disadvantages it presents and that have motivated the development of the alternative procedures discussed in the next two sections.

Acoustics problems are governed by the wave equation for the pressure field, $\bar{p}(\mathbf{x}, t)$. Expressing the pressure field as $\bar{p}(\mathbf{x}, t) = \mathrm{Re}\,[p(\mathbf{x})\mathrm{e}^{-\mathrm{j}\omega t}]$, with $\mathrm{j} = \sqrt{-1}$, this is transformed into the following Helmholtz equation:

$$\nabla^2 p + \kappa^2 p = 0, \quad \mathbf{x} \in \mathsf{V}, \tag{1}$$

where $\kappa = \omega/c$ is the wavenumber, ω is the frequency of vibration, c is the speed of sound, and V is the cavity domain. The solution of Eq. (1) may be expressed in terms of the Kirchhoff–Helmholtz boundary integral equation, once the Green function, G, has been determined. It is defined as the solution of the following fundamental problem:

$$\nabla^2 G + \kappa^2 G = -\delta(\mathbf{x} - \mathbf{x}_*), \tag{2}$$

where δ denotes the Dirac delta function, and $\mathbf{x}_*$ represents the unit source location. Boundary conditions must be specified both for Eqs. (1) and (2). In the first problem, Neumann boundary conditions, Dirichlet boundary conditions or combinations of them are applied, this depending upon the physical problem to deal with. On the contrary, boundary conditions for the fundamental problem may be defined independently of the particular physical problem analyzed, and their choice has a great influence on the complexity of the resulting boundary integral equation. For the Kirchhoff–Helmholtz problem the following free-space Dirichlet boundary conditions are applied:

$$G \to 0 \quad \text{as } \|\mathbf{x} - \mathbf{x}_*\| \to \infty,$$

and the resulting Green's function reads

$$G(\mathbf{x}, \mathbf{x}_*, \kappa) = \frac{\mathrm{e}^{\mathrm{j}\kappa r}}{4\pi r},$$

where $r = \|\mathbf{x} - \mathbf{x}_*\|$. Multiplying Eq. (1) by G, Eq. (2) by p and subtracting the second from the first, the Kirchhoff–Helmholtz boundary integral equation is derived as

$$p(\mathbf{x}_*, \kappa) = \int_{\mathsf{S}} \left(G \frac{\partial p}{\partial n} - p \frac{\partial G}{\partial n} \right) \mathrm{d}\mathsf{S}(\mathbf{x}), \tag{3}$$

where $\mathsf{S} = \partial \mathsf{V}$, and n denotes the arclength along the direction of the outward unit normal to the surface S.

This integral equation may be used in order to determine the acoustic free modes of vibration of the cavity. Indeed, if Ψ denotes a natural mode of vibration, we have $\partial\Psi/\partial n = 0$ and, therefore, it satisfies the following boundary integral equation [see Eq. (3)]:

$$\Psi(\mathbf{x}_*, \kappa) = -\int_{\mathsf{S}} \Psi \frac{\partial G}{\partial n} \, \mathrm{d}\mathsf{S}(\mathbf{x}). \tag{4}$$

The numerical solution of this boundary integral equation yields the free modes of vibration of the cavity, Ψ_i, with the associated wavenumbers, κ_i, and the corresponding frequencies of vibration, $\omega_i = c\kappa_i$. A number of numerical approaches may be applied for the discretization of Eq. (4). Whatever approach is considered, the discretized version of Eq. (4) will be of the type

$$\mathbf{H}(\kappa)\mathbf{\Psi} = \mathbf{0}, \tag{5}$$

where $\mathbf{H}$ is a square matrix with entries that are transcendental functions of the wavenumber, κ, whereas $\mathbf{\Psi}$ is a column matrix collecting the unknowns describing the shape of the acoustic modes; for instance, using a collocation method, $\mathbf{\Psi}$ collects the values of Ψ at the collocation points defined over S. The nontrivial solution of the eigenproblem in Eq. (5) identifies the cavity spectrum: the frequencies of vibration are obtained from the wavenumbers, κ_i, such that $\det[\mathbf{H}(\kappa_i)] = 0$, whereas the mode shapes are evaluated through the corresponding eigenvectors, $\mathbf{\Psi}_i$. Unfortunately, because of the transcendental dependence of $\det[\mathbf{H}(\kappa)]$ on κ, the evaluation of its roots is not an easy task, and becomes almost impossible if wavenumber ranges with a high density of roots occur (as it typically happens in cavity acoustics). This problem, that will be shown and discussed in Section 5 for a specific numerical test case, is the motivation for the development of alternative formulations yielding easier eigenproblems.

3. The Succi method

A formulation alternative to the Kirchhoff–Helmholtz approach has been introduced by Succi (1987) for the analysis of the acoustic field inside an automobile cabin. The main difference with respect to the method of Section 2 is in the definition of the Green function. Here, the solution of Eq. (2) is determined in an arbitrary domain, $\mathsf{V}_{\rm ext}$, enclosing the cavity under examination, with the application of homogeneous Neumann boundary conditions. In particular, following Succi (1987) the exterior arbitrary domain is chosen as a simple-shaped domain, as close as possible to the cavity, for which eigenfunctions and eigenvalues of the Laplace operator are known, for homogeneous Neumann boundary conditions. Then, the application of the eigenfunction method yields the following expression for the Green function used in this approach:

$$G(\mathbf{x}, \mathbf{x}_*, \kappa) = \sum_m \frac{\Phi_m(\mathbf{x})\Phi_m(\mathbf{x}_*)}{\kappa_m^2 - \kappa^2}, \tag{6}$$

where in the domain $\mathsf{V}_{\rm ext}$ and for homogeneous Neumann boundary conditions, $-\kappa_m^2$ is the mth eigenvalue of the Laplace operator and Φ_m is the corresponding eigenfunction. A second peculiar aspect of this methodology is that the unknown natural acoustic modes of vibration of the cavity are expressed in terms of the eigenfunctions of the Laplace operator in $\mathsf{V}_{\rm ext}$, as given by the following expansion:

$$\Psi(\mathbf{x}, \kappa) = \sum_m q_m(\kappa)\Phi_m(\mathbf{x}). \tag{7}$$

Substitution of Eqs. (6) and (7) into Eq. (4) yields

$$\sum_m q_m(\kappa)\Phi_m(\mathbf{x}_*) = -\sum_m \frac{\Phi_m(\mathbf{x}_*)}{(\kappa_m^2 - \kappa^2)} \sum_i q_i(\kappa) \int_{\mathsf{S}} \Phi_i \frac{\partial \Phi_m}{\partial n}\, \mathrm{d}\mathsf{S}(\mathbf{x}). \tag{8}$$

That is a boundary integral equation for the unknown cavity acoustic free vibration modes. Finally, observing that Φ_m are a set of linearly independent functions, the satisfaction of the equation above implies the satisfaction of the following set of algebraic equations:

$$[\mathbf{F} + \mathbf{\Lambda}]\mathbf{q} = \kappa^2 \mathbf{q}, \tag{9}$$

where the entries of the matrix $\mathbf{F}$ are given by

$$F_{mi} = \int_{\mathsf{S}} \Phi_i \frac{\partial \Phi_m}{\partial n}\, \mathrm{d}\mathsf{S}(\mathbf{x});$$

matrix $\mathbf{\Lambda}$ is a diagonal matrix defined as $\Lambda_{mi} = \kappa_m^2 \delta_{mi}$, with δ_{mi} denoting the Kronecker delta function, and $\mathbf{q}$ is the vector collecting the unknown coefficients of the eigenfunction expansion of the cavity acoustic free vibration modes.

Following the Succi method, identification of the cavity acoustic spectrum is much simpler than in the case of the method presented in Section 2. Indeed, it is reduced to the solution of the standard eigenproblem expressed in Eq. (9), with the eigenvalues of the matrix $[\mathbf{F} + \mathbf{\Lambda}]$ representing the (square) of the wavenumbers corresponding to the acoustic frequencies of vibration of the cavity, and the eigenvectors representing the components of the cavity acoustic free vibration modes along the function base defined by the Laplace operator eigenfunctions, Φ_m. However, an implicit inaccuracy is hidden in this methodology: the resulting acoustic free vibration modes do not satisfy the homogeneous Neumann boundary conditions in the cavity domain, since they are expressed in terms of functions that satisfy the homogeneous Neumann conditions only at the boundary of the exterior domain, $\mathsf{V}_{\rm ext}$. The level of inaccuracy depends upon the difference between the cavity and the exterior domain and, as will be shown in Section 5 for a specific numerical test case, barely decreases with the increase of the number of eigenfunctions used in the mode finite expansion. The methodology presented in the next section has been developed in order to overcome this inaccuracy, while maintaining the simplicity in the identification of the cavity spectrum as the solution of a standard eigenproblem.

4. The boundary-field integral approach

This methodology, inspired by the Succi approach (Succi, 1987), starts from the need of expressing the natural acoustic modes of vibration of the cavity in such a way that satisfaction of the homogeneous Neumann boundary conditions is guaranteed.

To this aim, first a set of functions, $\chi_m(\mathbf{x})$, linearly independent in the cavity domain, V, and having normal derivative equal to zero along ∂V, are introduced, and then the eigenfunction expansion in Eq. (7) is replaced by

$$\Psi(\mathbf{x},\kappa) = \sum_m q_m(\kappa)\chi_m(\mathbf{x}). \tag{10}$$

However, this choice implies that repeating the procedure of Section 3, i.e., substituting Eqs. (10) and (6) into Eq. (3), does not yield a boundary integral equation for the natural acoustic modes of vibration of the cavity that is satisfied [like Eq. (8)] by the solution of a standard algebraic eigenproblem. This occurs because of the fact that the functions used for the expansion of G [see Eq. (6)] are not equal to those used for the expansion of Ψ, and therefore the right- and left-hand side of the resulting boundary integral equation are not directly expressed as combinations of the same functions. In order to get a set of algebraic equations for the mode unknowns, q_m, it is necessary to project the boundary integral equation onto the mode base functions (Galërkin approach), but in this case the unknown wavenumbers would appear in each entry of the system matrix and their evaluation would not be reduced to the solution of a standard eigenproblem of the type of that in Eq. (9).

In this work, in order to yield a standard eigenproblem for the cavity acoustics starting from the function expansion in Eq. (10), the Green function has been chosen to be the fundamental solution of the Laplace operator, i.e.,

$$\nabla^2 G = -\delta(\mathbf{x} - \mathbf{x}_*). \tag{11}$$

Thus, the Green function does not depend on the wavenumber, κ, and multiplying Eq. (1) by G, Eq. (11) by p and subtracting the second from the first, yields the following integral equation for the acoustic natural modes of vibration:

$$\Psi(\mathbf{x}_*,\kappa) = -\int_{\mathsf{S}} \Psi \frac{\partial G}{\partial n}\,\mathrm{d}\mathsf{S}(\mathbf{x}) + \kappa^2 \int_{\mathsf{V}} \Psi G\,\mathrm{d}\mathsf{V}(\mathbf{x}), \tag{12}$$

where the disadvantage of the presence of a volume integral extended over the volume occupied by the cavity is joined to the advantage of having an explicit dependence on the frequency of vibration. The final step of the methodology consists of substituting Eq. (10) into Eq. (12), and then projecting onto the base functions, χ_m. This procedure yields the following standard eigenproblem:

$$\mathbf{V}^{-1}[\mathbf{S} + \mathbf{A}]\mathbf{q} = \kappa^2\mathbf{q}, \tag{13}$$

where $\mathbf{q}$ is an unknown vector that collects the coefficients that appear in the expansion of the cavity free vibration modes. In the equation above the entries of the matrix $\mathbf{S}$ are given by

$$S_{mi} = \int_{\mathsf{V}} \chi_m \chi_i\,\mathrm{d}\mathsf{V}(\mathbf{x}),$$

the entries of the matrix $\mathbf{A}$ are given by

$$A_{mi} = \int_{\mathsf{V}} \chi_i \left[\int_{\mathsf{S}} \chi_m \frac{\partial G}{\partial n}\,\mathrm{d}\mathsf{S}(\mathbf{x})\right]\mathrm{d}\mathsf{V}(\mathbf{x}_*),$$

whereas the entries of the matrix $\mathbf{V}$ are given by

$$V_{mi} = \int_{\mathsf{V}} \chi_i \left[\int_{\mathsf{V}} \chi_m G\,\mathrm{d}\mathsf{V}(\mathbf{x})\right]\mathrm{d}\mathsf{V}(\mathbf{x}_*)$$

[note that the matrix $\mathbf{S}$ would be a diagonal matrix, if the linearly independent functions used in Eq. (10) were orthogonal]. Therefore, from the evaluation of the eigenvalues and eigenvectors of the matrix $\mathbf{V}^{-1}[\mathbf{A} + \mathbf{S}]$ it is possible to determine the acoustic spectrum of the cavity examined.

Note that the structure of the solving system remains unchanged whatever the boundary conditions associated to the Green-function differential problem in Eq. (11). For instance, if free-space Dirichlet boundary conditions are applied, the Green function becomes

$$G(\mathbf{x},\mathbf{x}_*) = \frac{1}{4\pi r}, \tag{14}$$

whereas, if homogeneous Neumann conditions over the boundary of a volume, V_{ext}, enclosing the cavity are considered, the Green function is given by

$$G(\mathbf{x},\mathbf{x}_*) = \sum_m \frac{\Phi_m(\mathbf{x})\Phi_m(\mathbf{x}_*)}{\kappa_m^2}, \tag{15}$$

where $-\kappa_m^2$ is the mth eigenvalue of the Laplace operator and Φ_m is the corresponding eigenfunction in V_{ext}.

5. Numerical results

The numerical investigation starts with the spectrum of a simple-shaped cavity, for which the exact analytical solution is known, and compares it to the numerical solution obtained by each of the above three formulations. Next, the boundary-field integral approach is applied to fuselage-like and fairing-like cavity shapes of aeronautical interest.

The simple-shaped cavity considered is a rectangular box having square base with unit-length sides and height that is three times larger than the length of the base sides. For this cavity configuration, in Cartesian coordinates, the exact acoustic modes of vibration are expressed by

$$\Psi_{imn}(x,y,z) = \cos(\mathrm{i}\pi x)\cos(m\pi y)\cos(n\pi z/3), \tag{16}$$

whereas the exact frequencies of vibration correspond to the wavenumbers

$$\kappa^2_{imn} = (\mathrm{i}\pi)^2 + (m\pi)^2 + (n\pi/3)^2. \tag{17}$$

First, the Kirchhoff–Helmholtz boundary integral formulation is analyzed. The problem is discretized by using the expansion given in Eq. (10), with the unknown coefficients, q_m, composing the vector $\mathbf{\Psi}$ in Eq. (5), and the functions χ_m obtained as products of three orthogonal polynomials, each function of one of the three Cartesian coordinates, and satisfying homogeneous Neumann boundary conditions. These orthogonal polynomials, ξ_i, are determined from the following sequences of even and odd polynomials:

$$\xi^e_n(\eta) = \sum_{k=0}^{n} c^{(n)}_{2k}\eta^{2k}, \quad \xi^o_n(\eta) = \sum_{k=0}^{n+1} c^{(n)}_{2k+1}\eta^{2k+1}, \quad \{\xi_i\} = \{\xi^e_n\} \cup \{\xi^o_n\},$$

with $n = 0,\ldots,N$. All ξ^e_n polynomials are orthogonal to all ξ^o_n polynomials since the first ones are symmetric functions of η, whereas the others are anti-symmetric functions of η. The orthogonality condition with respect to the lower order polynomials of the same type in the domain $[-1,1]$, and the imposition of a null derivative in $\eta = 1$ yield the coefficients of the polynomials to be used in the definition of χ_m ($c^{(0)}_0$ and $c^{(0)}_1$ have been assumed to be equal to one). As already shown in Section 2, the Kirchhoff–Helmholtz boundary integral formulation predicts the acoustic natural frequencies of vibration as those corresponding to the values of the wavenumber, κ, for which $\det[\mathbf{H}(\kappa)] = 0$ [see Eq. (5)]. The values of $\det[\mathbf{H}(\kappa)]$ in the wavenumber range $0<\kappa<5$ that have been computed using an increasing number of Gaussian points for the integration of the entries of matrix $\mathbf{H}$ are shown in Fig. 1. In the same figure, on the zero-determinant axis the stars represent the exact values of the wavenumbers corresponding to the natural frequencies of vibration given by Eq. (17), for the indices i, m, n ranging from 0 to 1 (note that only two polynomials per spatial direction are used in the definition of the χ_m functions, and natural frequencies of vibration coming from higher index modes would not be predictable). These results show that, as expected, the zeros of the determinant of $\mathbf{H}(\kappa)$ tend to the analytical natural frequencies as the accuracy of the numerical integration increases. However, when the zeros tend to concentrate into a narrow region, it is very difficult to capture their values since the determinant becomes an almost-zero highly wavy function. This is illustrated in Fig. 2, where a zoom of the convergence analysis in the wavenumber region occupied by the frequencies of vibration higher than the first one is depicted. Thus, these results demonstrate that the Kirchhoff–Helmholtz boundary integral formulation is not a reliable tool for identifying acoustic spectra.

Next, the Succi method presented in Section 3 is examined. Initially, for the definition of the Green function, an outer domain having the same shape of the cavity, but with double-length sides is considered. The numerical investigation is focused on the eigenvalues, κ^2, corresponding to the wavenumber range $0<\kappa<4$, with the aim of comparing the exact frequencies of vibration corresponding to the wavenumbers given by Eq. (17), with the eigenvalues of the system matrix $[\mathbf{F}+\mathbf{\Lambda}]$ arising in Succi's approach. This analysis is depicted in Fig. 3, where the stars represent the eigenvalues calculated from the Succi method for a different number, N_{modes}, of eigenfunctions in the expansion of the Green function [see Eq. (6)]. These results show that, in the wavenumber region considered, increasing the accuracy of G yields a too high number of numerical eigenvalues that do not tend to get closer to the exact ones. Indeed, it can be observed that the eigenvalues predicted by this method are closer to those of the outer domain where the expansion eigenfunctions have been defined (the squares in Fig. 3), than to those of the cavity (the crosses in Fig. 3). Note that in the wavenumber range considered, the number of outer-domain eigenvalues is higher than that of the cavity analyzed because of the larger dimensions of the outer-domain. Thus, better results may be obtained if the outer domain gets closer to the cavity examined. This is illustrated in Fig. 4, where the same convergence analysis is performed considering an outer domain having the same shape of the cavity, but with dimensions $[1.5\times1.5\times3.5]$. Although the number of the numerical eigenvalues in $0<\kappa^2<16$ exceeds that of the exact ones, their number is lower than in the previous case, and a better agreement with the analytical results may be observed. Nevertheless, the values of the numerical eigenvalues remain quite different from the values of those given by Eq. (17). As observed in Section 3, this inaccuracy should be

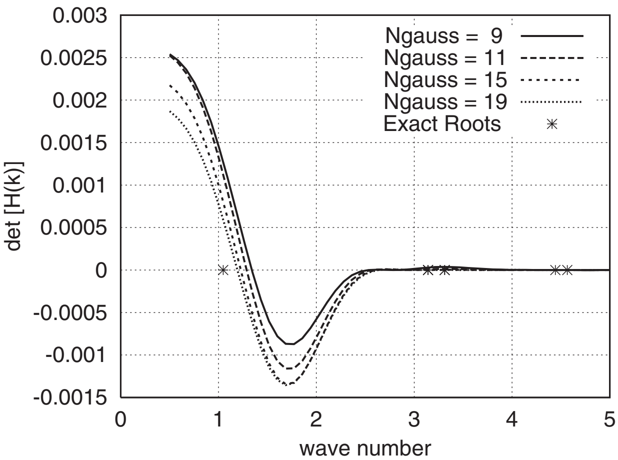

Fig. 1. Convergence behavior of det[$\mathbf{H}(\kappa)$] in terms of the number of quadrature Gaussian points that are used in the evaluation of integrals.

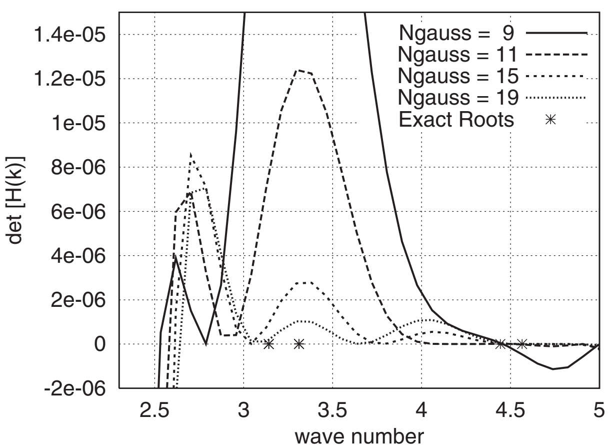

Fig. 2. Convergence behavior of det[$\mathbf{H}(\kappa)$] in terms of the number of quadrature Gaussian points that are used in the evaluation of integrals (zoom).

caused by the fact that the cavity free-vibration modes expressed by Eq. (7) do not satisfy the homogeneous Neumann conditions at the cavity boundary, even increasing the number of eigenfunctions used for the definition of G and Ψ. This is shown in Figs. 5 and 6 that depict, respectively, the cavity vibration mode corresponding to the first frequency of vibration, and the cavity vibration mode corresponding to the second frequency of vibration, along the central axis of the cavity. Specifically, the analytical acoustic modes of vibration given by Eq. (16) are compared with those predicted numerically for an increasing number of eigenfunctions, Φ_m, in the expansion of Ψ_1 and Ψ_2. These figures show that, although with the increase of N_{modes} two extremes of the numerical acoustic modes of vibration tend to get closer to the cavity boundary, they do not respect the homogeneous Neumann conditions at the boundary of the cavity (represented by the two internal vertical lines) even for $N_{\text{modes}} = 30$ (of course, the numerical modes given by Eq. (7) satisfy this condition at the boundary of the outer domain, that in Figs. 5 and 6 is represented by the two external vertical lines).

Then, the boundary-field integral approach is applied in order to assess its capability to predict the correct cavity spectrum. In this analysis, the expansion of the numerical acoustic modes of vibration, Ψ_m, is accomplished by using the same set of orthogonal polynomials used in the Kirchhoff–Helmholtz formulation. First, the results obtained using the Green function expressed in Eq. (15), with the eigenfunctions Φ_m corresponding to the outer domain used for the results of Fig. 3 are presented. They are shown in Fig. 7 for an increasing number of eigenfunctions and demonstrate that the

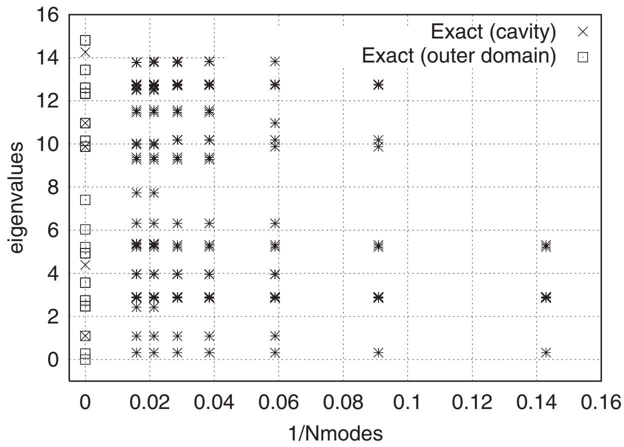

Fig. 3. Eigenvalues from Succi's method: convergence behavior in terms of the number of expansion eigenfunctions that are used both in Eqs. (6) and (7). Outer domain dimensions [$2 \times 2 \times 6$].

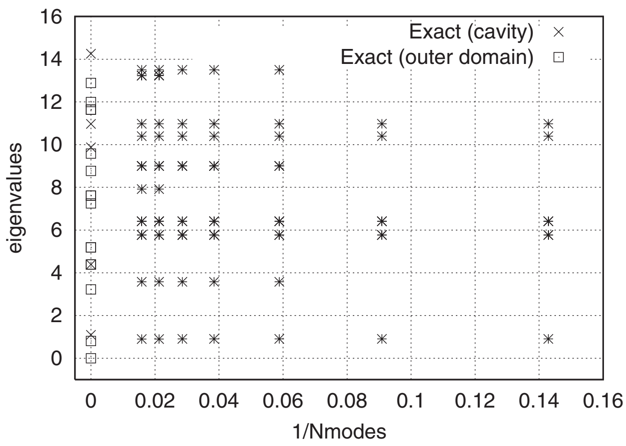

Fig. 4. Eigenvalues from Succi's method: convergence behavior in terms of the number of expansion eigenfunctions that are used both in Eqs. (6) and (7). Outer domain dimensions [$1.5 \times 1.5 \times 3.5$].

numerical eigenvalues converge to the analytical ones and no additional (spurious) eigenvalues are found in the range $0 < \kappa^2 < 16$. Of course, an even better convergence behavior is obtained if an outer domain closer to the cavity is considered. This is shown in the results of Fig. 8, where the convergence analysis is performed using the Green function expressed through the eigenfunctions of the outer domain of dimensions [$1.5 \times 1.5 \times 3.5$]. Figs. 7 and 8 demonstrate that the numerical integration of the approach based on the Green function of Eq. (15) converges quite rapidly. For these results, a fixed number of Gaussian points has been used that is sufficiently large to assure satisfactory accuracy. The Green function given by Eq. (14) has been applied to obtain the results presented in Fig. 9. In this case, the convergence analysis is performed by increasing the number of Gaussian points used for numerical integration. Fig. 9 confirms that this methodology is able to predict the correct eigenvalues of the cavity acoustics. It is also worth noting that, in this case, the numerical acoustic modes match the analytical ones.

Finally, the boundary-field integral approach is validated for cavities that can be of interest in aeronautical applications. First, the simulation of the acoustic problem inside the fuselage of an airplane is examined by considering the spectrum of a cylindrical cavity. In analogy with the case analyzed above, the cylinder examined has a unit radius and length that is three times larger than the radius. Also in this case, the analytic evaluation of the acoustic spectrum is available for comparison. Using cylindrical coordinates, (x, θ, r), the exact acoustic modes of vibration are expressed by

$$\Psi_{imn}(x, \theta, r) = \cos(\mathrm{i}\pi x/3)\cos(m\theta)\mathrm{J}_m(\gamma_{mn} r), \tag{18}$$

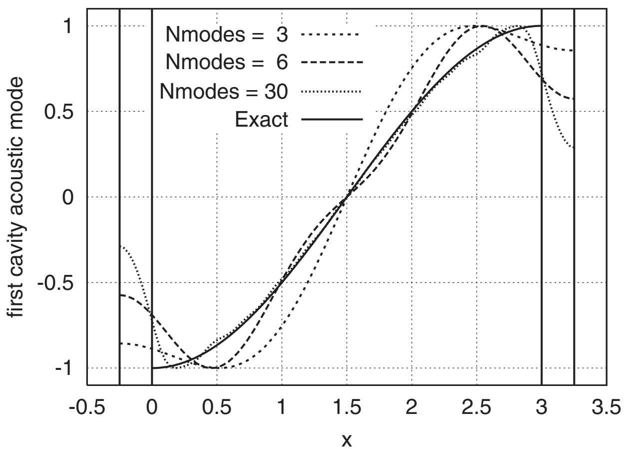

Fig. 5. First acoustic mode of vibration of the cavity from Succi's method: convergence behavior in terms of the number of eigenfunctions that are used for its finite expansion in Eq. (7). Outer domain dimensions [1.5 × 1.5 × 3.5].

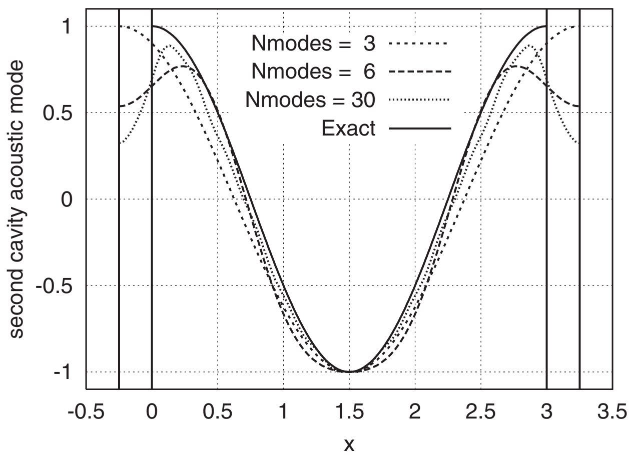

Fig. 6. Second acoustic mode of vibration of the cavity from Succi's method: convergence behavior in terms of the number of eigenfunctions that are used for its finite expansion in Eq. (7). Outer domain dimensions [1.5 × 1.5 × 3.5].

whereas the exact frequencies of vibration correspond to the wavenumbers

$$\kappa_{imn}^2 = (\mathrm{i}\pi/3)^2 + (\gamma_{mn})^2, \tag{19}$$

where J_m denotes the Bessel function of the first kind of order m, whereas γ_{mn} denotes the nth root of the first derivative of J_m. The exact spectrum of the cylinder is compared in Fig. 10 with the results obtained through the boundary-field integral approach that applies the free-space Green's function. Akin to the box-shaped cavity, as the number of Gaussian points used for numerical integration increases, the boundary-field integral approach predicts a spectrum that converges to the exact one. For this cavity, relating the index m to the three-index system, ijk, the functions χ_m used in the eigenfunction expansion have been obtained through the following expression:

$$\chi_{ijk}(x,\theta,r) = \xi_i(2x/L)\cos(\mathrm{j}\theta)\left\{\begin{array}{c}\xi_k^e(r/R)\\ \xi_k^o(r/R)\end{array}\right\}, \tag{20}$$

for $x \in [-L/2, L/2]$, $\theta \in [0,\pi]$ and $r \in [-R, R]$. In the equation above, L indicates the height of the cavity, R denotes the radius of the cavity and the polynomials used in the radial direction are of even type if the azimuthal index is even and are of odd type if the azimuthal index is odd (this assures continuity of the function along any circumferential

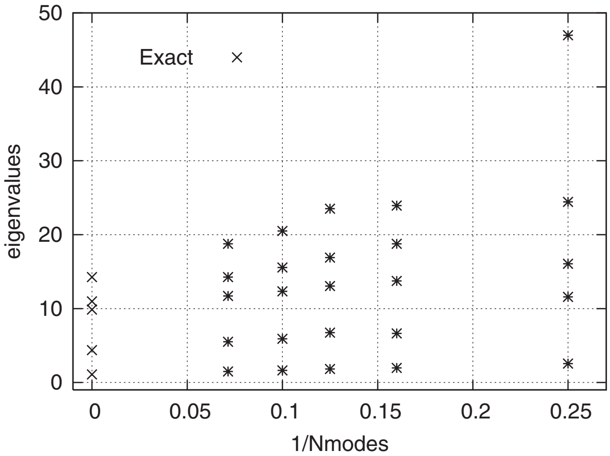

Fig. 7. Eigenvalues from the boundary-field integral method: convergence behavior in terms of the number of expansion eigenfunctions that are used in the expression of G. Outer domain dimensions $[2 \times 2 \times 6]$.

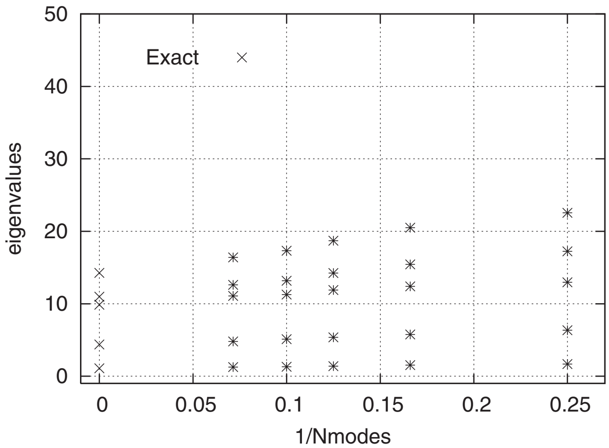

Fig. 8. Eigenvalues from the boundary-field integral method: convergence behavior in terms of the number of expansion eigenfunctions that are used in the expression of G. Outer domain dimensions $[1.5 \times 1.5 \times 3.5]$.

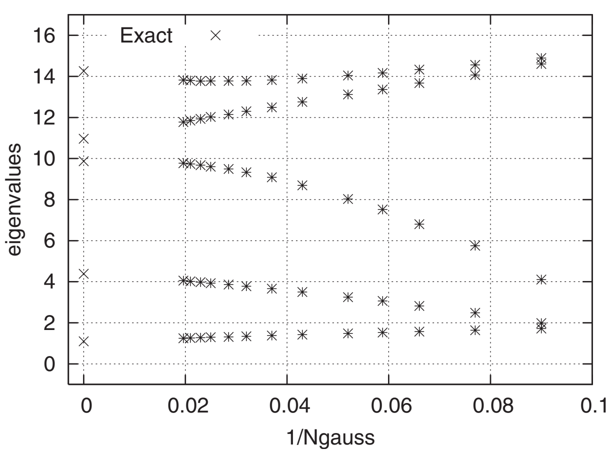

Fig. 9. Eigenvalues from the boundary-field integral method with free-space Green's function: convergence behavior in terms of the number of quadrature Gaussian points.

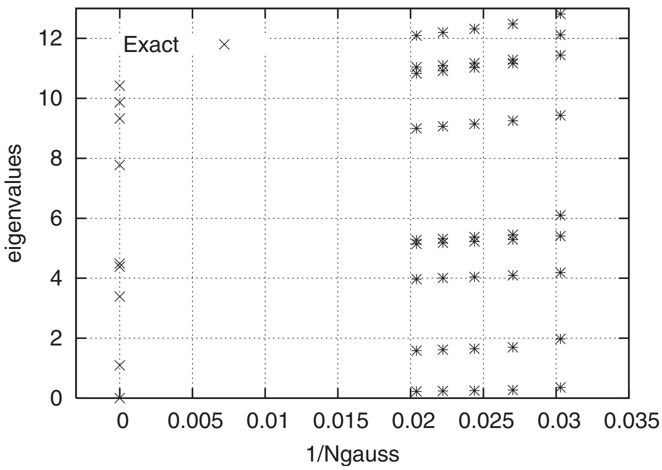

Fig. 10. Eigenvalues of a cylindrical cavity from the boundary-field integral method with free-space Green's function: convergence behavior in terms of the number of quadrature Gaussian points.

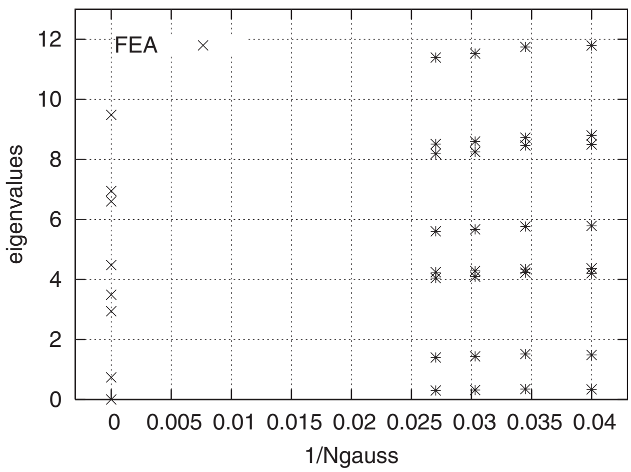

Fig. 11. Eigenvalues of a fairing-shaped cavity from the boundary-field integral method with free-space Green's function: convergence behavior in terms of the number of quadrature Gaussian points.

coordinate). Note that, Eq. (20) is applicable for the identification of the functions χ_m for any convex cavity that may be mapped into a cylindrical domain (with the introduction of a suited set of curvilinear coordinates such that each subdomain of the boundary surface is a portion of a coordinate surface $\eta = \text{const}$, with the η coordinate line locally orthogonal to it).

The last case examined concerns a fairing-like cavity. It is defined by adding to one end of the cylindrical cavity considered above a semi-sphere of unit radius and removing the upper 10% of the spherical dome. Such a cavity is mapped into a cylindrical domain and Eq. (20) is used for the definition of the functions χ_m. In this case, the spectrum is not evaluated analytically, and the results from the boundary-field integral approach (with the free-space Green's function) are correlated with the converged spectrum determined by a finite-element approach (FEA) analysis. This is shown in Fig. 11, where the convergence behavior confirms the capability of the boundary-field integral approach to predict the acoustic spectrum in more complex cavities.

Note that a procedure similar to that used in the last application could be applied for the identification of the functions χ_m, in case of arbitrarily shaped cavities. Specifically, it would consist of first mapping the shape cavity into a spherical domain, followed by the definition of the functions χ_m as products of periodic functions along the coordinate surfaces with Bessel functions in the radial direction.

6. Conclusions

The theoretical derivation of a nonstandard integral formulation for the solution of the Helmholtz equation has been presented. For the acoustic modal identification of a resonating cavity of arbitrary shape it yields a standard eigenvalue problem. At the cost of a volume integral evaluation, it avoids the difficulty related to the root-finding of a transcendental equation that typically arises in boundary integral approaches for cavity acoustics. For a simple box-shaped cavity, numerical results have confirmed the capability of the proposed methodology to circumvent the uncertainties related to the determination of the eigensolution of the classical Kirchhoff–Helmholtz equation. In addition, the numerical investigation presented has demonstrated that the present approach avoids the inaccuracy appearing in the eigenvalue-problem formulation introduced earlier by Succi (1987), in that it yields a solution that converges to the exact one with respect to the discretization parameters introduced.

The capability of the boundary-field integral approach to predict the acoustic spectrum has been confirmed for more complex cavity shapes. In particular, the spectrum given by the proposed methodology converges to that obtained analytically for a cylindrical cavity, and converges to that obtained through a finite-element approach for a fairing-like-shaped cavity.

A critical issue in the application of the present boundary-field integral approach is the determination of the set of linearly independent functions used in the expansion of the acoustic eigenfunctions. In this work, procedures for their identification in case of box-shaped cavities and cavities that may be mapped into a cylindrical domain have been described and applied. A similar procedure is also suggested for the case of a cavity having arbitrary shape.

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